ON SAMPLING THEOREMS FOR NON BANDLIMITED SIGNALS

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Abstract.[1] It is well-known that certain non bandlimited signals such as splines can be reconstructed from uniformly spaced samples similar to bandlimited signals. This usually requires noncausal IIR filters. We revisit this result and consider extensions such as derivative sampling theorems and pulse sampling theorems. It turns out that spline-like signals can often be reconstructed from joint sampling of amplitude and derivative using only FIR filters. We also briefly consider discrete time versions of these results.

1. INTRODUCTION

Consider a continuous-time signal which can be modelled in the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi(t-k) \qquad (1)$$

where $\phi(t)$ is a fixed function. Such signals arise in many situations. For example, when $\phi(t)$ is the sinc function $\sin \pi t/\pi t$, $x(t)$ is a bandlimited (π-BL) signal and is infinitely differentiable. When $\phi(t)$ is a B-spline function, $x(t)$ is a spline with limited differentiability everywhere. For fixed $\phi(t)$, the space of signals of the form (1) where $\{c_n\} \in \ell_2$ will be called S_ϕ. Bandlimited signals and splines are examples of such spaces; under some conditions on $\phi(t)$, a multiresolution analysis can be generated [1].

Suppose $\phi(t)$ has the zero-crossing or **Nyquist(1)** property $\phi(n) = \delta(n)$. In this case $x(n) = c_n$, and

$$x(t) = \sum_n x(n)\phi(t-n),$$

so $x(t)$ can be recovered from the samples $x(n)$. This happens for example when $\phi(t) = \sin \pi t/\pi t$. For more general $\phi(t)$ such as splines, this zero crossing property is not true, but it still turns out to be possible to recover $x(t)$ from the samples $x(n)$ [1], [9] even though $x(t)$ may not be bandlimited. The basic idea is that the samples have the form $x(n) = \sum_k c_k \phi(n-k)$ which is a discrete time convolution. We can therefore recover $\{c_k\}$ from $\{x(n)\}$ using the digital filter $1/\Phi_d(z)$ where

$$\Phi_d(z) = \sum_n \phi(n) z^{-n} \qquad (2)$$

[1]Work supported in part by the ONR grant N00014-99-1-1002, USA.

provided it is realizable. The preceding can be regarded as a uniform sampling theorem for signals of the form (1), which in general are not bandlimited. For the case where $\phi(t)$ is a spline, $\Phi_d(z)$ is FIR and has zeros both inside and outside the unit circle [5]. In order to ensure stability, $1/\Phi_d(z)$ has been implemented as a noncausal IIR filter. This idea works well for finite length signals like images. Unser, et al. not only advanced this technique, but have shown many applications [6]. Some variations of these sampling theorems have already been reported in the past [2]. In this paper we shall consider further new variations.

2. DERIVATIVE SAMPLING

Suppose we have available both $x(t)$ and the first derivative $\dot{x}(t)$. The samples of these obtained at **half the rate** are given by

$$x(2n) = \sum_k c_k \phi(2n-k) \text{ and } \dot{x}(2n) = \sum_k c_k \dot{\phi}(2n-k)$$

Evidently the total number of samples per unit time is unity as before. We can regard these samples as the outputs of the two channel filter bank shown in Fig. 1 where the analysis filters are

$$H_0(z) = \sum_n \phi(n) z^{-n}, \qquad H_1(z) = \sum_n \dot{\phi}(n) z^{-n}.$$

Under some conditions on these filters, we can recover c_n from $x(2n)$ and $\dot{x}(2n)$ perfectly. Once this is done, $x(t)$ can be reconstructed from (1). An advantage is that we can often obtain FIR reconstruction (i.e., make $F_0(z), F_1(z)$ FIR) as we shall demonstrate. Derivative sampling theorems can be useful, for example, when we have measurements of position and velocity of a moving target or car.

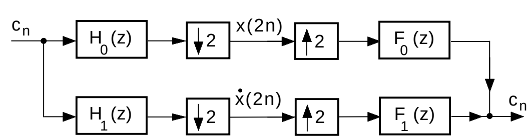

Fig. 1. Two channel filter bank for derivative sampling.

For example consider the case where $\phi(t)$ is the quadratic B-spline given by

$$b_2(t) = \begin{cases} t^2/2 & 0 \le t < 1 \\ 3/4 - (t-3/2)^2 & 1 \le t < 2 \\ (t-3)^2/2 & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\phi(1) = \phi(2) = 0.5$ and $\phi(n) = 0$ otherwise, so that $\Phi_d(z) = 0.5z^{-1}(1+z^{-1})$. Recovery of c_n from the full-rate samples $x(n)$ requires the IIR filter $2z/(1+z^{-1})$ which in addition is unstable (pole at $z = -1$). Now consider derivative sampling. We have

$$H_0(z) = 0.5z^{-1}(1 + z^{-1}).$$

The quadratic spline is continuously differentiable once, and the samples of the result are

$$\dot{\phi}(1) = -\dot{\phi}(2) = 1, \quad \text{and} \quad \dot{\phi}(n) = 0 \text{ otherwise},$$

so that

$$H_1(z) = z^{-1}(1 - z^{-1}).$$

The synthesis filters which give perfect reconstruction are uniquely given by

$$F_0(z) = z(1 + z) \quad \text{and} \quad F_1(z) = z(1 - z)/2 \tag{3}$$

These are simple FIR filters indeed!

3. SAMPLING AN N-TH ORDER SPLINE

There is excellent literature on splines [4]–[6] which allows us to generalize the preceding idea for splines of arbitrary orders. In this section we show how.

3.1. Review Of Splines

The Nth order B-spline (with integer knots), denoted $b_N(t)$, is merely the result of convolving the unit pulse $p(t)$ with itself N times (Fig. 2). It is therefore a finite duration signal, nonzero only in $0 < t < N + 1$ and has the Fourier transform

$$B_N(j\Omega) = e^{-j\Omega(N+1)/2}\Big(\frac{\sin(\Omega/2)}{\Omega/2}\Big)^{N+1} \tag{4}$$

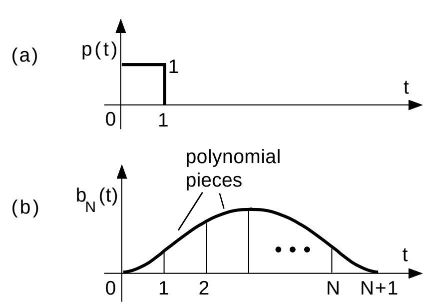

Fig. 2. (a) The pulse $p(t)$ and (b) the Nth order B-spline $b_N(t)$ generated by convolving $p(t)$ with itself N times.

Between any two integers, $b_N(t)$ is a polynomial of order N. These are joined at the integers such that $b_N(t)$ is continously differentiable everywhere $N - 1$ times; moreover the Nth derivative is a piecewise constant. The closed form expression for $b_N(t)$ is

$$b_N(t) = \sum_{k=0}^{N+1} \binom{N+1}{k} (-1)^k \frac{(t-k)^N}{N!} \mathcal{U}(t-k) \tag{5}$$

where $\mathcal{U}(t)$ is the unit-step. Consider a function $x(t)$ of the form (1) where $\phi(t) = b_N(t)$. Then $x(t)$ has two properties: (i) it is a polynomial between integers, and (ii) it is continuously differentiable $N - 1$ times. A function with these two properties is called an Nth order spline.

There is a famous result [4] which says that any Nth order spline can be expressed as in (1) with $\phi(t)$ chosen as the B-spline $b_N(t)$. As explained in Sec. 1, the spline $x(t)$ can be reconstructed from its samples $x(n)$ as long as the digital filter $1/\Phi_d(z)$ is realizable. From its definition we see that $b_N(t)$ is symmetric with respect to its midpoint, so the FIR filter $\Phi_d(z) = \sum_{n=1}^{N} b_N(n)z^{-n}$ has zeros in reciprocal pairs $(z_0, 1/z_0)$. This means that $1/\Phi_d(z)$ is unstable if implemented causally. This motivated noncausal implementations, which turn out to be quite efficient [5].

3.2. Derivative Sampling For Splines

Imagine now that instead of sampling $x(t)$ at the integers, we sample $x(t)$ and its $N - 1$ derivatives at a rate N times smaller. Then the total number of samples per unit time is still the same as before. However, it now turns out that $x(t)$ can be recovered from these samples using digital filters that are not only stable but in fact **FIR**. To prove this note that the samples of the kth derivative, $x^{(k)}(Nn)$, can be written using (1) as

$$x_k(n) \stackrel{\Delta}{=} x^{(k)}(Nn) = \sum_i c_i b_N^{(k)}(Nn - i), 0 \le k \le N - 1$$

where the superscripts (k) denote the kth derivative. Now $b_N^{(k)}(i)$ are nonzero only for $1 \le i \le N$; define

$$H_k(z) = \sum_{i=0}^{N} b_N^{(k)}(i)z^{-i} \tag{6}$$

Then the samples $x_k(n)$ are the outputs of the N band maximally decimated FIR analysis bank shown in Fig. 3. Since the sample $b_N^{(k)}(0) = 0$ (see Fig. 2), we can write $H_k(z) = z^{-1}\widehat{H}_k(z)$ where $\widehat{H}_k(z)$ is causal FIR with length $\le N$. The polyphase matrix [7] corresponding to $\{\widehat{H}_k(z)\}$ is therefore a constant (as in a transform coder). *If this matrix is nonsingular*, then there exists an FIR perfect-reconstruction synthesis bank with filters $\widehat{F}_k(z) = \sum_{n=-(N-1)}^{0} \widehat{f}_k(n)z^{-n}$. The synthesis bank corresponding to $\{H_k(z)\}$ is then $\{z\widehat{F}_k(z)\}$. In short, we have an FIR PR filter bank as shown in Fig. 3 where

$$H_k(z) = \sum_{n=0}^{N} h_k(n)z^{-n}, \quad F_k(z) = \sum_{n=-N}^{0} f_k(n)z^{-n}$$

with $h_k(0) = f_k(0) = 0$. This shows that c_n can be reconstructed from the derivative samples $x^{(k)}(Nn)$.

It is insightful to express the spline $x(t)$ directly in terms of the samples of the derivatives. For this note that $c_n = \sum_k \sum_i x_k(i) f_k(n-Ni)$. Substituting into the spline model (1) and simplifying, we get

$$x(t) = \sum_{k=0}^{N-1} \sum_i x^{(k)}(Ni) s_k(t - Ni) \tag{7}$$

where $s_k(t) = \sum_n f_k(n)\phi(t-n)$. We can regard $\{s_k(t)\}$ to be a bank of continuous-time reconstruction filters.

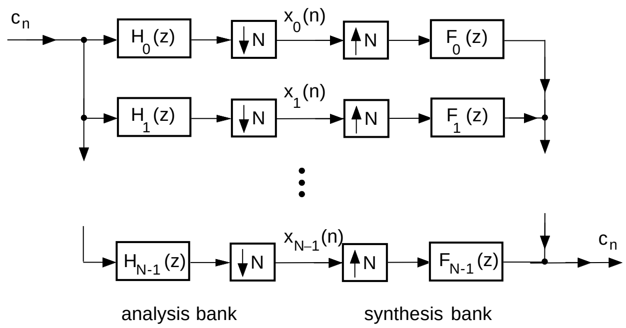

Fig. 3. The N channel analysis bank model for the samples of the derivatives $x_k(n) = x^{(k)}(Nn)$, and the synthesis bank to reconstruct c_n from the derivative samples.

3.3. Examples

Let us revisit the example of quadratic splines, where $N = 2$ and the synthesis filters are as in (3). We have

$$x(t) = \sum_i x(2i)s_0(t-2i) + \sum_i \dot{x}(2i)s_1(t-2i)$$

where $s_0(t)$ and $s_1(t)$ are given by

$$\begin{aligned} s_0(t) &= \phi(t+1) + \phi(t+2) \\ s_1(t) &= 0.5\phi(t+1) - 0.5\phi(t+2). \end{aligned}$$

As a second example consider cubic splines which can be represented as in (1) where $\phi(t) = b_3(t)$. Using (5), or the fact that $b_3(t) = b_2(t) * p(t)$, we can obtain the samples of $b_3(t)$ and its derivatives and obtain the analysis filters $H_k(z)$ in (6). Since $b_3(t)$ is supported in $0 < t < 4$, these are FIR, and

$$\begin{aligned} H_0(z) &= z^{-1}(1+4z^{-1}+z^{-2})/6 \\ H_1(z) &= z^{-1}(1-z^{-2})/2 \\ H_2(z) &= z^{-1}(1-2z^{-1}+z^{-2}) \end{aligned}$$

Using standard techniques [7] the synthesis filters for perfect reconstruction are

$$\begin{aligned} F_0(z) &= z(1+z+z^2) \\ F_1(z) &= z(1-z^2) \\ F_2(z) &= z(2-z+2z^2)/6 \end{aligned}$$

The reconstruction functions $s_k(t) = \sum_n f_k(n)\phi(t-n)$ can now be obtained. The result is

$$\begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1/3 & -1/6 & 1/3 \end{bmatrix} \begin{bmatrix} \phi(t+1) \\ \phi(t+2) \\ \phi(t+3) \end{bmatrix}$$

Having found $s_k(t)$, the spline $x(t)$ can be recovered from the samples $x(3n)$, $\dot{x}(3n)$ and $\ddot{x}(3n)$ using (7).

4. DISCRETE TIME CASE

As mentioned in Sec. 1 signals which can be represented by the model of (1) can be recovered from appropriately sampled versions even though they may not be bandlimited. The discrete time analog of this is also known [8]. One simple example is a signal $x[n]$ that can be modelled as the output of an interpolation filter (Fig. 4). In the following discussion, we use the polyphase representation $F(z) = \sum_{i=0}^{M-1} z^i R_i(z^M)$ for convenience. Note that $R_i(z) = [z^{-i}F(z)]_{\downarrow M}$, that is, $r_i(n) = f(Mn-i)$.

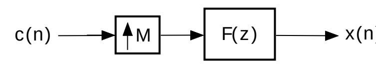

Fig. 4. Signal model allowing reconstruction from samples.

Since $x(n) = \sum_k c(k)f(n-Mk)$ this is analogous to (1) which was obtained by superposition of uniform shifts of $\phi(t)$. One would then expect that $x(n)$ can be recovered from the M-fold decimated version $x(Mn)$. Indeed we have

$$x(Mn) = \sum_k c(k)f(M(n-k)) = \sum_k c(k)r_0(n-k)$$

where $r_0(n) = f(Mn)$. This shows that $c(n)$ can be recovered as the output of $1/R_0(z)$ in response to the input $x(Mn)$. Thus, from $x(Mn)$ we can find $c(n)$, and use $x(n) = \sum_k c(k)f(n-Mk)$ to find $x(n)$.

This idea succeeds as long as $R_0(z)$ has no unit circle zeros. If this is not the case, we can look for another polyphase component $R_i(z)$ with this property. Then $x(Mn-i) = \sum_k c(k)f(M(n-k)-i) = \sum_k c(k)r_i(n-k)$ which shows that $c(k)$ can be recovered from the samples $x(Mn-i)$ by filtering through $1/R_i(z)$. If none of the polyphase components is free from unit circle zeros, then we can try other combinations of samples. Thus let $M = 2$ and

$$F(z) = 1 + z - z^2 + z^3 = 1 - z^2 + z(1+z^2)$$

which shows $R_0(z) = 1 - z$ and $R_1(z) = 1 + z$. Both of these have unit circle zeros. So we cannot recover $x(n)$ from the two-fold decimated version $x(2n)$ or from $x(2n-1)$ in a stable manner. Now consider the signal $x(n)$ and its first difference $x(n) - x(n-1)$. If we decimate these by **four** we get

$$x_0(n) = x(4n), \quad x_1(n) = x(4n) - x(4n-1).$$

Together, these two signals still imply an average two-fold decimation. We will show that $x(n)$ can be recovered from $x_0(n)$ and $x_1(n)$ using stable, in fact FIR, filters. This is analogous to the derivative sampling scheme of Sec. 2. To prove the preceding claim note that the signal model is $X(z) = C(z^2)F(z)$ so that

$$\begin{aligned} X_0(z) &= \Big[C(z)[F(z)]_{\downarrow 2}\Big]_{\downarrow 2} \\ X_1(z) &= \Big[C(z)\Big[(1-z^{-1})F(z)\Big]_{\downarrow 2}\Big]_{\downarrow 2} \end{aligned}$$

Defining $H_0(z) = [F(z)]_{\downarrow 2} = 1 - z$ and

$$H_1(z) = [(1 - z^{-1})F(z)]_{\downarrow 2} = -2z,$$

we see that the samples $x_0(n)$ and $x_1(n)$ can be represented as the outputs of an analysis bank (Fig. 5). Using the synthesis filters

$$F_0(z) = 1, \quad F_1(z) = -(1 + z^{-1})/2$$

we verify that this is a perfect reconstruction system. In short, $c(n)$ can be recovered from the samples $x(4n)$ and $x(4n) - x(4n-1)$ using the FIR filters $F_0(z)$ and $F_1(z)$. The original signal $x(n)$ can then be recovered from the basic model of Fig. 4.

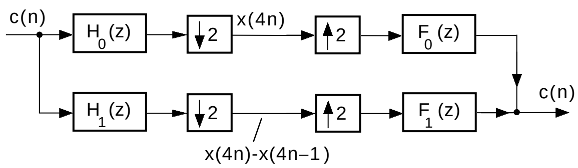

Fig. 5. Two channel filter bank for difference-sampling.

5. CONCLUDING REMARKS

In Sec. 3.1 we introduced derivative sampling of splines, and showed that for quadratic and cubic splines we can always reconstruct the spline coefficients c_n with FIR filters. We have not proved this formally for arbitrary order splines. In attempting such a proof, it would be helpful to note that the Nth order spline and its $N-1$ derivatives are supported in $0 < t < N+1$ so that there are at most N nonzero integer-samples. The problem then reduces to proving analytically that a certain constant polyphase matrix [7] is nonsingular.

Similar remarks apply for the discrete time model of Sec. 4. Thus, assume that the model filter $F(z)$ is FIR. Given M and $F(z)$, can we show that there exists an appropriate extension of the M-fold decimation (like the difference sampling of Sec. 4) from which $x(n)$ can be recovered using FIR filters alone? This remains an open problem as well. In attempting this, it might be useful to notice that FIR reconstruction from *nonuniformly* decimated versions have been shown to be possible under some conditions [8].

A final remark we wish to make relates to pulse sampling. The idea has its origin in a homework problem in [3] pertaining to Shannon sampling of bandlimited signals. Consider a continuous time signal $x(t)$ bandlimited, say, to π. It can be recovered from integer samples $x(n)$ using lowpass filtering (sinc interpolation). Such sampling can be regarded as multiplication of $x(t)$ with an impulse train. Suppose a practical sampling system performs multiplication with a pulse train as shown in Fig. 6 where $p(t)$ is periodic but not quite an inpulse train. Expressing $p(t)$ as a Fourier series, it can be shown that we can indeed recover $x(t)$ from $x(t)p(t)$ for almost any $p(t)$. This follows from the fact that multiplication with $p(t)$ is equivalent to replicating the Fourier transform $X(j\Omega)$ around the harmonics of $p(t)$. This creates no aliasing as long as the pulse rate is large enough, and we can isolate one copy by filtering.

A similar result holds for the more general signal model $x(t) = \sum_{k=-\infty}^{\infty} c_k \phi(t-k)$ because

$$y(t) = x(t)p(t) = \sum_{k=-\infty}^{\infty} c_k \phi(t-k)p(t-k)$$

where we have exploited the pulse periodicity $p(t) = p(t-k)$. Thus $y(t) = \sum_{k=-\infty}^{\infty} c_k \phi_1(t-k)$, where $\phi_1(t) = \phi(t)p(t)$ so that

$$\begin{aligned} Y(j\Omega) &= \sum_{k=-\infty}^{\infty} c_k \int \phi_1(t-k)e^{-j\Omega t}dt \\ &= \sum_{k=-\infty}^{\infty} c_k e^{-j\Omega k}\Phi_1(j\Omega) \\ &= C(e^{j\Omega})\Phi_1(j\Omega) \end{aligned}$$

We can recover c_k using $C(e^{j\Omega}) = Y(j\Omega)/\Phi_1(j\Omega)$ under some obvious realizability conditions. Once $\{c_k\}$ is known, $x(t)$ is fully determined.

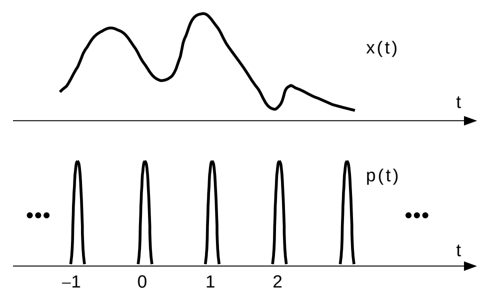

Fig. 6. Pulse-sampling of a continuous-time signal $x(t)$.

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