Coherent Integration Loss Due to White Gaussian Phase Noise

Mark A. Richards, *Senior Member, IEEE*

***Abstract*—We develop a simple analytic expression for the change in coherent weighted integration gain due to a white Gaussian error or noise in the phase of the integrated samples. Our expression is shown by simulation to be very accurate for any reasonable value of phase noise standard deviation. The result is useful in estimating the performance impact on coherent signal processing systems of oscillator noise, residual motion compensation errors, and other system imperfections that are manifested primarily as phase errors.**

***Index Terms*—Coherent integration, integration loss, phase noise, radar, sonar.**

I. Derivation of Coherent Integration Error

CONSIDER a series of complex data samples $x_n = Ae^{j(\phi+\phi_n)}$ where A and ϕ are constants, but the sequence $\{\phi_n\}$ are independent and identically distributed (i.i.d.) zero-mean Gaussian random variables with standard deviation σ. We consider $Ae^{j\phi}$ to be the desired measurement and ϕ_n to be a phase error. Phase errors frequently arise from phase deviations in the local oscillators of radar, sonar, and communication systems, uncompensated sensor motion errors, and other sources. The Gaussian model for phase errors is a common assumption [1], and it has been shown that oscillator phase errors are in fact Gaussian under widely applicable assumptions [2]. Note that the power (magnitude-squared) of x_n is A^2. A weighted coherent sum of N such data values is formed as

$$w = \sum_{n=1}^{N} a_n x_n \tag{1}$$

where the $\{a_n\}$ are known deterministic weights. Equation (1) can model a wide variety of signal processing algorithms that involve coherent integration. Examples include coherent detection based on multiple samples, clutter cancellation, linear filtering, and computation of the discrete Fourier transform. The power in w is the random variable

$$\begin{aligned} z =& |w|^2 = ww^* \\ =& \sum_n Aa_n e^{j(\phi+\phi_n)} \sum_m Aa_m^* e^{-j(\phi+\phi_m)} \\ =& A^2 \sum_n \sum_m a_n a_m^* e^{j(\phi_n-\phi_m)}. \end{aligned} \tag{2}$$

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The author is with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250 USA (e-mail: mark.richards@ece.gatech.edu).

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We are interested in estimating the change in the power of the coherently integrated and detected sample z due to the phase noise. This change is a significant performance metric in many systems, e.g., radar moving target indication [3], [4]. The power in the coherent sum in the absence of phase noise ($\phi_n \equiv 0$) is just $A^2 \sum_n \sum_m a_n a_m^* = A^2 |\sum_n a_n|^2$. In the coherent integration case when $a_n \equiv 1 \; \forall n$, this simplifies to $N^2 A^2$. As another example, a radar clutter canceller always has $\sum_n a_n \equiv 0$, so the power is zero in the absence of phase noise.

When phase noise is present, z is a random variable, and we must compute its expected value

$$\begin{aligned} E\{z\} =& A^2 \sum_n \sum_m a_n a_m^* E\left\{e^{j(\phi_n-\phi_m)}\right\} \\ =& A^2 \sum_n \sum_m a_n a_m^* \Psi_{nm} \end{aligned} \tag{3}$$

where

$$\Psi_{nm} \equiv E\left\{e^{j(\phi_n-\phi_m)}\right\}.$$

Because the phase noise samples $\{\phi_n\}$ are i.i.d. and stationary, we have

$$\Psi_{nm} = \begin{cases} 1, & m = n \\ E\left\{e^{j\phi_n}\right\} E\left\{e^{-j\phi_m}\right\} = \Phi\Phi^* = |\Phi|^2, & m \neq n \end{cases} \tag{4}$$

where

$$\Phi \equiv E\left\{e^{j\phi_n}\right\}. \tag{5}$$

At this point, our problem has reduced to computing Φ.

To proceed, we need to use the specific model of the probability density function (pdf) of ϕ_n, which we denote $p_\phi(\phi_n)$. We use a zero-mean Gaussian with standard deviation σ

$$p_\phi(\phi_n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\phi_n^2/2\sigma^2}. \tag{6}$$

Strictly speaking, this is not possible: a valid pdf for a phase measurement must be limited to the interval $[0, 2\pi)$. However, if the standard deviation is small, then $p_\phi(\phi_n)$ will approach zero as ϕ_n approaches $\pm\pi$, and the Gaussian is a useful model. In practice, we are interested in standard deviations of only a few degrees, so that $\pm\pi$ corresponds to many standard deviations out on the tail of the Gaussian distribution.

By definition, we have for $m \neq n$

$$\begin{aligned} \Phi =& E\left\{e^{j\phi_n}\right\} \equiv \int_{-\pi}^{\pi} e^{j\phi_n} p_\phi(\phi_n)\, d\phi_n \\ \cong& \int_{-\infty}^{\infty} e^{j\phi_n} p_\phi(\phi_n)\, d\phi_n \end{aligned} \tag{7}$$

where the approximation relies again on the variance of the Gaussian being small, so that the pdf is effectively nonzero only in a small region around $\phi = 0$. Substituting in the definition of the Gaussian pdf, we obtain

$$\Phi = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(j\phi_n - (1/2\sigma^2)\phi_n^2)} d\phi_n. \quad (8)$$

Integral number 3.323–2 in [5] is

$$\int_{-\infty}^{\infty} e^{qx - p^2x^2} dx = \frac{\sqrt{\pi}}{p} e^{q^2/4p^2}. \quad (9)$$

By identifying

$$p = \frac{1}{\sigma\sqrt{2}};\ q = j;\ x = \phi_n \quad (10)$$

we can put (8) into the form of (9) and conclude that

$$\Phi = \frac{1}{\sigma\sqrt{2\pi}} e^{j^2/4(1/2\sigma^2)} \frac{\sqrt{\pi}}{\left(\frac{1}{\sigma\sqrt{2}}\right)} = e^{-\sigma^2/2} \quad (11)$$

and

$$|\Phi|^2 = e^{-\sigma^2}. \quad (12)$$

We can now return to Ψ_{mn}, which becomes

$$\Psi_{nm} = \begin{cases} 1, & m = n \\ e^{-\sigma^2}, & m \neq n. \end{cases} \quad (13)$$

Finally, the mean power of the coherently integrated measurement with phase noise is

$$E\{z\} = A^2 \left[\sum_{n=1}^{N} |a_n|^2 + e^{-\sigma^2} \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m \neq n}}^{N} a_n a_m^* \right]. \quad (14)$$

This is the desired result. It is easily computed given the phase noise variance σ^2 and the weighting coefficients $\{a_n\}$.

The case where $a_n \equiv 1$ for all n is of special interest; this is the model for coherent integration of samples for detection, for example. It is common to compute the integration loss L in this case, defined as the ratio of the power when phase noise is present to the power when it is not

$$L = \frac{E\{z\}}{A^2 \left| \sum_{n=1}^{N} a_n \right|^2} = \frac{\sum_{n=1}^{N} |a_n|^2 + e^{-\sigma^2} \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m \neq n}}^{N} a_n a_m^*}{\left| \sum_{n=1}^{N} a_n \right|^2} \text{ dB}. \quad (15)$$

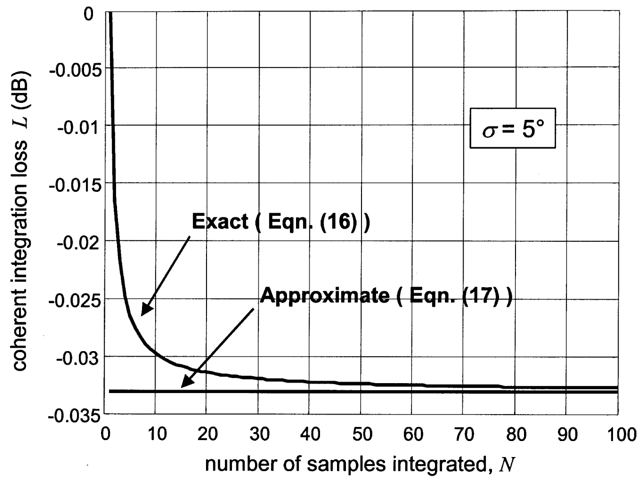

Fig. 1. Comparison of "exact" formula for the change in integrated power due to phase noise (16) to the simplified approximation (17) as a function of the number of samples integrated N. $\sigma = 5°$.

With $a_n \equiv 1$, (15) simplifies to

$$L = \left[\frac{N + e^{-\sigma^2} N(N-1)}{N^2} \right] = \left[\frac{1 + e^{-\sigma^2}(N-1)}{N} \right]. \quad (16)$$

If N is large, we can further simplify (16) to

$$L \cong e^{-\sigma^2}, \qquad N \gg 1. \quad (17)$$

Finally, this approximation is especially simple when expressed in decibels

$$L \cong 10 \log_{10} \left[e^{-\sigma^2} \right] = -4.343\sigma^2 \text{ dB}, \qquad N \gg 1. \quad (18)$$

The basic result of (14) can be used for calculations other than integration loss. For example, a two-pulse radar clutter canceller can be modeled by choosing $N = 2$, $a_1 = +1$, and $a_2 = -1$. The power computed using (14) is then the limit on clutter attenuation due to phase noise. Approximating $\exp(-\sigma^2)$ by the first two terms of its power series gives the classical approximations for clutter attenuation found in [3] and [4] (after adjusting for their use of real, rather than complex, signals). Our result is applicable over a larger range of phase variance.

Fig. 1 shows the difference in the coherent integration loss formula (16) for L and the large-N approximation of (17) as a function of the number of samples integrated N for the specific example of $\sigma = 5°$. The difference in the predicted mean loss is less than 0.01 dB for $N > 4$ samples. As the phase variance rises, the value of N for which the error in the approximation is less than 0.01 dB also rises, reaching $N = 14$ for $\sigma = 10°$ and $N = 55$ for $\sigma = 20°$. However, these are quite large values of phase error; the approximation of (17) or (18) is very good for a small number of samples integrated and reasonably small values of phase error.

II. Validation by Simulation

A simple MATLAB Monte Carlo simulation has been implemented to validate the estimated loss of (16). A sequence of unit

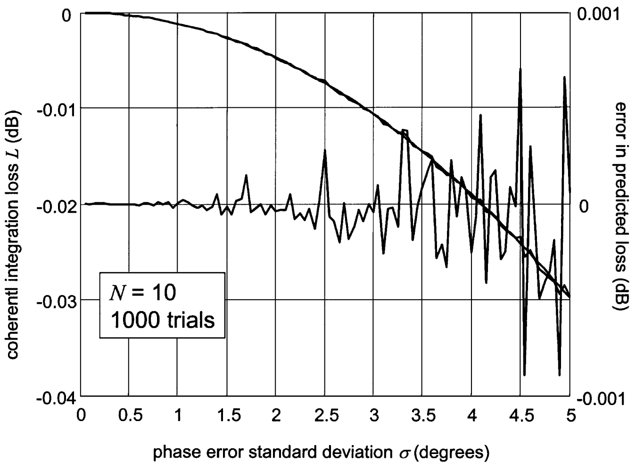

Fig. 2. Agreement between simulated and analytically estimated (16) coherent integration loss due to phase noise. $N = 10$ samples integrated, 1000 Monte Carlo trials. (Left ordinate) Simulated and predicted integration loss are nearly indistinguishable. (Right ordinate) Difference in the two curves. For this case, it is well under 0.001 dB.

amplitude ($A = 1$) complex phasors with a zero-mean Gaussian phase is generated. The samples are then numerically integrated (summed). The magnitude-squared of the sum is computed, and then normalized by the power without phase noise and converted to decibels. Fig. 2 compares this Monte Carlo estimate of L against the analytically predicted value given in (16) for $N = 10$ samples integrated and 1000 Monte Carlo trials averaged. We see that the analytical prediction is an excellent match to the simulated data. The difference between the two curves in decibels is plotted against the right ordinate and is well under 0.001 dB for this case. For larger N, the variance of the integration loss is less and the curves are an even closer fit. The error between simulated and predicted loss for $N = 10$ does not exceed 0.01 dB until the phase noise variance is about 20°, a large value. At this latter level of phase variance, the "3-sigma" points are $\pm 60^\circ$, and the approximations to the phase pdf and the change of integration limits from $\pm\pi$ to $\pm\infty$, which both relied on a "narrow Gaussian" pdf, are becoming invalid.

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